Entire functions of exponential type, almost periodic in Besicovitch's sense on the real hyperplane

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Abstract

Suppose that an almost periodic in Besicovitch's sense function $f(x)$ of several variables is the restriction to the real hyperplane of an entire function of exponential type b. Then its spectrum is contained in the ball of radius b with the center in the origin.

In his paper [3] H. Bohr showed that the spectrum of an almost periodic function $f(x)$ on the real axis $\mathbb{R}$ is a subset of $[-\sigma, \sigma]$, as long as f is the restriction to $\mathbb{R}$ of an entire function of an exponential type σ. R. Boas [2] extended the assertion to almost periodic functions on $\mathbb{R}$ in Besicovitch's metric (for brevity, B-almost periodic functions). In the general case, these functions are unbounded on $\mathbb{R}$, hence the proof of the latter assertion is more difficult.

H. Bohr's result was generalized to almost periodic functions in a finite dimensional space by S.Yu. Favorov and O.I. Udodova [8]. But the case of B–almost periodicity is more complicated, because restrictions to straight lines of B–almost periodic functions in $\mathbb{R}^p$ are not necessary almost periodic.

It should be mentioned that B–almost periodic functions of several variables were considered earlier in [4], [6], [7]. But in [6], [7] the spectrum of functions was not under consideration, and in [4] the author studied only B–almost periodic functions with bounded Besicovitch's norm in a tube domain with a cone in the base.

Our proof differs from ones in [2], [3], [8] and is based on estimates of entire functions and Logvinenko's theorem [9] on the growth of entire functions of several variables on the hyperplane $\mathbb{R}^p$.

We will use the following notations.

By $z = x + iy$, $z = (z_1, ..., z_p)$, $x = (x_1, ..., x_p)$, $y = (y_1, ..., y_p)$ we denote the vectors in $\mathbb{C}^p$ (or, respectively, in $\mathbb{R}^p$, $'x$ means the vector $(x_2, ..., x_p) \in \mathbb{R}^{p-1}$, $\langle x, y\rangle$ is the inner product in $\mathbb{R}^p$. Next, $|z|$, $|x|$, $|'x|$ are the Euclidean norms in the spaces $\mathbb{C}^p$, $\mathbb{R}^p$, and $\mathbb{R}^{p-1}$, respectively. By dx, $d'x$, and dx_1 we denote the Lebesgue measure in $\mathbb{R}^p$, $\mathbb{R}^{p-1}$, and $\mathbb{R}$, respectively. Furthermore, $B(x, \delta)$ means the open ball in $\mathbb{R}^p$ of radius δ with the center in x, C with lower indexes are constants, depending only on f.

Besicovitch's norm of a locally integrable function $f(x)$ in $\mathbb{R}^p$ is the limit

$$\|f\|_B = \overline{\lim_{T\to\infty}} \left(\frac{1}{2T}\right)^p \int\limits_{[-T,T]^p} |f(x)| dx.$$

The function $f(x)$ is called *B–almost periodic* in $\mathbb{R}^p$, if for any $\varepsilon > 0$ there is a (generalized) trigonometric polynomial

$$P(x) = \sum c_n e^{i\langle x, \lambda^{(n)}\rangle}, \quad c_n \in \mathbf{C},\ \lambda^{(n)} \in \mathbf{R^p}, \tag{1}$$

such that

$$\|f - P\|_B < \varepsilon.$$

The Fourier coefficient of f is the limit

$$a(\lambda, f) = \lim_{T\to\infty} \left(\frac{1}{2T}\right)^p \int\limits_{[-T,T]^p} f(x) e^{-i\langle x,\lambda\rangle} dx. \tag{2}$$

The spectrum $\mathtt{sp} f$ of $f(x)$ is the set

$$\{\lambda \in \mathbf{R^p} : a(\lambda, f) \neq 0\}.$$

Note that the existence the limit (2) and countability of the spectrum follow easily from the definition of B–almost periodicity, and the equality

$$a(\lambda, P) = \begin{cases} c_n, & \lambda = \lambda^{(n)} \\ 0, & \lambda \neq \lambda^{(n)}, \end{cases} \tag{3}$$

which holds for any n and any polynomial (1).

By [7], for any B–almost periodic function there exists a sequence of polynomial (1) (the so-called Bochner-Fejer sums) with $\lambda^n \in \mathtt{sp}f$, which approximate f.

The main result of our paper is the following theorem.

Theorem 1. *Let a B–almost periodic function $f(x)$ in $\mathbb{R}^p$ extends to $\mathbb{C}^p$ as an entire function with the bound*

$$|f(z)| \le C_0 e^{\sigma|z|}. \tag{4}$$

Then we have $\mathrm{sp}f \subset B(0,\sigma)$.

The proof of this theorem is based on the following statement.

Theorem 2. *Let $f(x), x \in \mathbb{R}^p$ be a function with a finite norm $\|f\|_B$. If f can be extended to $\mathbb{C}^p$ as an entire function with estimate (4), then*

$$|f(x)| \le C_1 \prod_{j=1}^{p} (1+|x_j|)^p \qquad \forall x \in \mathbb{R}^p. \tag{5}$$

We get the proof of theorem 2, using the following auxiliary results.

Theorem A ([9])**.** *Let $f(z)$ be an entire function on $\mathbb{C}^p$, which satisfies (4), and E be a δ-net in $\mathbb{R}^p$. If $\sigma\delta < K(p)$, then*

$$\sup_{x\in R^p} |f(x)| \le (1-\sigma\delta)^{-1} \sup_{x\in E} |f(x)|.$$

Theorem B (see, for example, [5], p. 311)**.** *Let a function $g(w)$ be a holomorphic in $\mathbb{C}^+ = \{w \in \mathbb{C}, \mathrm{Im}\, w > 0\}$, continuous in the closure of $\mathbb{C}^+$, and satisfy the estimate*

$$|g(w)| \le ce^{a|w|},\ w \in \mathbb{C}^+. \tag{6}$$

If

$$\int_{-\infty}^{+\infty} \frac{\log^+ |g(t)|}{1+t^2} dt < \infty, \tag{7}$$

then

$$\log|g(w)| \le \frac{\mathrm{Im} w}{\pi} \int_{-\infty}^{+\infty} \frac{\log|g(\mathrm{Re} w + t)|}{t^2 + (\mathrm{Im} w)^2} dt + h\mathrm{Im} w,\ \ w \in \mathbb{C}^+,$$

where $h = \overline{\lim_{t\to+\infty}} \frac{\log|g(it)|}{t}$.

In the case $\sup_{\mathbb{R}} |g(w)| < \infty$ theorem B yields the well known version of Fragment–Lindelöf Principle

$$|g(w)| \le \sup_{\operatorname{Im} w=0} |g(w)| e^{h \operatorname{Im} w}, \quad w \in \mathbb{C}^+. \tag{8}$$

Proof of Theorem 2. Since $\|f\|_B < \infty$, we get for any $\delta \in (0,1)$ and $\widetilde{x} \in \mathbb{R}^p$

$$\int\limits_{B(\widetilde{x},\delta)} |f(x)| dx \le \int\limits_{[-\widetilde{x}-\delta, \widetilde{x}+\delta]^p} |f(x)| dx \le C_2 \left(1 + |\widetilde{x}|\right)^p .$$

Therefore there is a constant $C_3 < \infty$ such that for any ball of radius δ there exists a point x' in the ball with

$$|f(x')| \le C_3 \delta^{-p} \left(1 + |x'|\right)^p .$$

Put

$$g(z) = f(z) \prod_{j=1}^{p} \left(\frac{\sin z_j}{z_j} \right)^p .$$

We have

$$|g(z)| \le C_0 e^{\left(\sigma + p^2\right)|z|}, \quad z \in \mathbb{C}^p.$$

Since $|g(x)| \le C_4 \delta^{-p}$ at the points of the δ-net, we see that theorem A with a suitable δ implies the bound

$$\sup_{x \in \mathbb{R}^p} |g(x)| \le C_5. \tag{9}$$

Using (9), we apply inequality (8) first in the domain $\mathrm{Im} z_1 > 0$, and then in the domain $\mathrm{Im} z_1 < 0$. We get

$$|g(z_1, {}'x)| \le C_5 e^{\left(\sigma + p^2\right)|y_1|} \tag{10}$$

for all $z_1 = x_1 + iy_1 \in \mathbb{C}$, ${}'x \in \mathbb{R}^{p-1}$. Apply (8) to the function $g(z_1, \dots, z_p)$ as a function in the variable z_2 and use (10) instead of (9). Repeating these arguments by the variables $z_3, \dots, z_p$, we get

$$|g(z)| \le c_5 e^{\left(\sigma + p^2\right)(|y_1| + \ldots + |y_p|)}.$$

Therefore,

$$\left| f(z) \cdot \prod_{j=1}^{p} \left(\frac{\sin z_j}{z_j} \right)^p \right| \le C_6$$

on the set $A = \{z \in \mathbf{C^p} : |y_1| \le 1, \dots |y_p| \le 1\}$. Hence,

$$\left| f(z) \cdot \prod_{j=2}^{p} \left(\frac{\sin z_j}{z_j} \right)^p \right| \le C_7 (1 + |z_1|)^p$$

on the set $\{z : z \in A, z \notin \bigcup_n B(n\pi, \frac{1}{2})\}$. By the Maximum Principle, we get the same inequality with the constant $2^p C_7$ instead of C_7 at every point of A. Repeating these arguments $p-1$ times, we obtain (5). Theorem is proved.

For the proof of Theorem 1 we need the following Lemma.

Lemma. *Suppose that $f(z)$ is an entire function in $\mathbb{C}^p$, which satisfies (4), and its restriction to $\mathbb{R}^p$ satisfies the condition $\|f\|_B \le \infty$. Then for any $s_0 \in (0, \infty)$, $T \ge T(s_0)$, and $s \in (0, s_0)$ we get*

$$\int_{[-T,T]^p} |f(x_1 + is, 'x)| e^{-s\sigma} dx \le C_8 T^p,$$

with $C_8 = 2^{p+1}(1 + 2 \cdot 3^p \|f\|_B)$.

Proof of the Lemma. By theorem 2, the function $f(z_1, 'x)$ satisfies (7) in the variable z_1 for any fixed $'x \in \mathbb{R}^{p-1}$. Taking into account (4), we get

$$\overline{\lim_{t \to \infty}} \frac{\log |f(iy_1, \ 'x)|}{y_1} \le \sigma.$$

Hence Theorem B implies for any $s > 0$

$$\log |f(x_1 + is, \ 'x)| \le \frac{s}{\pi} \int_{-\infty}^{+\infty} \frac{\log |f(t + x_1, \ 'x)| dt}{t^2 + s^2} + \sigma s.$$

Since the measure $\frac{s}{\pi} \cdot \frac{dt}{t^2+s^2}$ is a probability one on $\mathbb{R}$, we get for any locally integrable function $h(t)$ on $\mathbb{R}$

$$\exp \left(\frac{s}{\pi} \int_{|t| \le 2T} \frac{h(t) dt}{t^2 + s^2} \right) \le \frac{s}{\pi} \int_{|t| \le 2T} \frac{e^{h(t)} dt}{t^2 + s^2} + \frac{s}{\pi} \int_{|t| > 2T} \frac{dt}{t^2 + s^2}.$$

Next,

$$\int\limits_{[-T,T]^p} |f(x_1+is,\ 'x)|e^{-\sigma s}dx \le \int\limits_{[-T,T]^p} \exp\left\{\frac{s}{\pi}\int\limits_{-\infty}^{+\infty}\frac{\log|f(t+x_1,\ 'x)|}{t^2+s^2}dt\right\}dx$$

$$\le \int\limits_{[-T,T]^p} \exp\left\{\frac{s}{\pi}\int\limits_{|t|\le 2T}\frac{\log|f(t+x_1,\ 'x)|}{t^2+s^2}dt\right\}\times$$

$$\exp\left\{\frac{s}{\pi}\int\limits_{|t|>2T}\frac{\log|f(t+x_1,\ 'x)|}{t^2+s^2}dt\right\}dx \le$$

$$\int\limits_{[-T,T]^p}\left(\frac{s}{\pi}\int\limits_{|t|\le 2T}\frac{|f(t+x_1,\ 'x)|}{t^2+s^2}dt+1\right)\exp\left\{\frac{s}{\pi}\int\limits_{|t|>2T}\frac{\log|f(t+x_1,\ 'x)|}{t^2+s^2}dt\right\}dx.$$

By (5) we have for $x\in[-T,\ T]^p$

$$\frac{s}{\pi}\int\limits_{|t|\ge 2T}\frac{\log|f(t+x_1,\ 'x)|}{t^2+s^2}dt \le \frac{s}{\pi}\int\limits_{|t|\ge 2T}\frac{C_3\left(\sum\limits_{j=1}^{p} p\log^+(1+|x_j|)+p\log^+ t\right)}{t^2+s^2}dt$$

$$\le C_3\left[p^2\log(1+T)\frac{s}{\pi}\int\limits_{|t|\ge 2T}\frac{dt}{t^2+s^2}+\frac{ps}{\pi}\int\limits_{|t|\ge 2T}\frac{\log^+ t dt}{t^2+s^2}\right].$$

Note that the latter expression bounds from above by $\log 2$ for $s\le s_0$ and $T\ge T(s_0)$.

Therefore, taking into account the inequality

$$\int\limits_{[-T,T]^p}|f(x)|dx \le 2\|f\|_B\,(2T)^p\,,\quad T\ge C_9$$

we obtain

$$\int\limits_{[-T,T]^p}|f(x_1+is,\ 'x)|e^{-s\sigma}dx \le 2\left(\frac{s}{\pi}\int\limits_{|t|\le 2T}\frac{dt}{t^2+s^2}\int\limits_{[-T,T]^p}|f(x_1+t,\ 'x)|dx+(2T)^p\right)$$

$$\leq 2^{p+1}T^p + 2\frac{s}{\pi} \int\limits_{|t|\leq 2T} \frac{dt}{t^2+s^2} \left[\int\limits_{[-3T,3T]} |f(x)|dx \right] \leq 2^{p+1}(1+2\cdot 3^p \|f\|_B)T^p.$$

Lemma is proved.

Proof of Theorem 2

Let A be an orthogonal matrix in $\mathbb{R}^p$. It is easy to check the equality

$$a(\lambda, f) = a(A^{-1}\lambda, f_A), \tag{11}$$

where $f_A(x) = f(Ax)$ and λ is an arbitrary vector in $\mathbb{R}^p$.

Indeed, it follows from (3) that this equality is true for any polynomial P (1). To prove it for an arbitrary B–almost periodic function, we can approximate it by polynomial P such that $\|f - P\|_B < \varepsilon$. Therefore, $\|f_A - P_A\|_B < K^p\varepsilon$, where $K = \max\limits_j |Ae_j|$, e_j is the natural basis in $\mathbb{R}^p$. Hence, we obtain (11).

Take $\lambda \in \mathbb{R}^p$, $|\lambda| > \sigma$. Since bound (4) is the same for f_A, we may suppose that $\lambda = (-\sigma - \eta, 0, \dots, 0)$, $\eta > 0$. In this case we have

$$a(\lambda, f) = \lim_{T\to\infty} \left(\frac{1}{2T}\right)^p \int\limits_{[-T,T]^p} f(x_1, {}'x) e^{ix_1(\sigma+\eta)} dx. \tag{12}$$

The function $f(z_1, x')$ is holomorphic in z_1, therefore for any $y_1 > 0$ we have

$$\int\limits_{[-T,T]^p} f(x_1, {}'x) e^{ix_1(\sigma+\eta)} dx =$$

$$i\int\limits_0^{y_1} \int\limits_{[-T,T]^{p-1}} f(-T+is, {}'x) e^{-iT(\sigma+\eta)-s(\sigma+\eta)} d\,{}'x\; ds +$$

$$\int\limits_{[-T,T]^p} f(x_1+iy_1, {}'x) e^{ix_1(\sigma+\eta)-y_1(\sigma+\eta)} dx -$$

$$i\int\limits_0^{y_1} \int\limits_{[-T,T]^{p-1}} f(T+is, {}'x) e^{iT(\sigma+\eta)-s(\sigma+\eta)} d\,{}'x\; ds$$

$$= I_1(T, y_1) + I_2(T, y_1) - I_3(T, y_1).$$

By Lemma, we get

$$|I_2(T, y_1)| \leq C_8 T^p e^{-\eta y_1},$$

hence for a given $\varepsilon > 0$ and sufficiently large y_1

$$\overline{\lim_{T\to\infty}} |(2T)^{-p} I_2(T, y_1)| \leq \varepsilon. \tag{13}$$

Next, $|I_1 - I_3| \leq G(T)$, where

$$G(x_1) = \int\limits_0^{y_1} \int\limits_{[-T,T]^{p-1}} e^{-s\sigma} \left(|f(x_1 + is,\ {}'x)| + |f(-x_1 + is,\ {}'x)|\right)\ ds\ d\,{}'x.$$

By Lemma, the Lebesgue measure of the set

$$E = \left\{ x_1 : \frac{T}{2} < |x_1| < T,\ G(x_1) > 3C_8 T^{p-1} |y_1| \right\}$$

is at most

$$\frac{1}{2C_8 |y_1| T^{p-1}} \int\limits_E G(x_1) dx_1 \leq$$

$$\frac{1}{3C_8 |y_1| T^{p-1}} \int\limits_0^{y_1} \int\limits_{[-T,T]^p} e^{-s\sigma} \left(|f(x_1 + is,\ {}'x)| + |f(-x_1 + is,\ {}'x)|\right) dx\ ds < \frac{2T}{3}.$$

Hence for some $T' \in \left[\frac{T}{2}, T\right] \setminus E$ we get $G(T') \leq 3C_8 |y_1| T^{p-1}$. Therefore we have

$$\overline{\lim_{T'\to\infty}} (2T')^{-p} |(I_1(T', y_1) - I_3(T', y_1))| = 0.$$

Thus, the latter bound and (13) yield $a(\lambda, f) = 0$. Theorem is proved.

References

[1] Besicovitch A.S. Almost periodic functions. - Cambridge university press, 1932. - 253 p.

[2] R. Boas R. P. Jr. Functions of exponential type. I. Duke Math. J. - 1944. - P. 9-15.

[3] Bohr H. Zur Theorie der fastperiodishen Functionen. III Teil. Dirichletentwich-lung analytisher Functionen. Acta Math. - 1926. V. 47, P. 237-281.

[4] Girya N.P. Almost periodic in Besicovitch's metric functions with the spectrum in a cone. // Matematichni Studii. - 2007. - V. 27., No. 2. - P. 163-173.

[5] Levin B. Ya. Distribution of zeros of entire functions. - M. - GITTL. - 1956. - 632 p.

[6] O.I. Udodova. Holomorphic almost periodic functions in various metrics. // Vestnik of Kharkov National University. Ser. "Mathematics, Applied Mathematics, and Mechanics". - 2003, V. 52. - No. 582. - P. 90-107.

[7] O.I. Udodova. Fourier series of holomorphic almost periodic in Besicovitch's sense functions.Vestnik of Kharkov National University. Ser. "Mathematics, Applied Mathematics, and Mechanics". - 2004. - V. 53. - No. 645. - P. 53-64

[8] Favorov S.Yu., Udodova O.I. Almost periodic functions in finite-dimensional space with the spectrum in a cone // Math.Phisics, Analys, Geometry. - 2002. - V. 9, No. 3. - P. 465-477

[9] Logvinenko V. N. On one multidimensional generalization Cartwright's Theorem // DAN SSSR. - 1974. V. 219, No. 3., P. 546-549.

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